Exact static solutions in Einstein-Maxwell-Dilaton gravity with arbitrary dilaton coupling parameter*

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Abstract

We present solution generating methods which allow us to construct exact static solutions to the equations of four-dimensional Einstein-Maxwell-Dilaton gravity starting with arbitrary static solutions to the pure vacuum Einstein equations, Einstein-dilaton or Einstein-Maxwell equations.

In four dimensions the field equations of the Einstein-Maxwell-dilaton gravity with arbitrary dilaton coupling parameter α, can be obtained from the following Einstein frame action [1]

$$\mathcal{A} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\partial_\mu\varphi\partial^\mu\varphi - e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu}\right) . \tag{1}$$

Here R is the Ricci scalar with respect to the space-time metric $g_{\mu\nu}$ (with a signature $(-,+,+,+)$), φ is the dilaton field and $F_{\mu\nu} = (dA)_{\mu\nu}$ is the Maxwell two-form.

The aim of the present work is to present methods for generating exact static solutions to the EMd-gravity equations generalizing the previous work of the author [2].

For a static space-time the metric can be written in the form

$$ds^2 = -e^{2u}dt^2 + e^{-2u}h_{ij}dx^i dx^j \tag{2}$$

where $e^{2u} = -g(\xi,\xi)$.

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In order to simplify calculations we will consider the pure electric case. In this case the Maxwell two-form is:

$$F = e^{-2u}\xi \wedge d\Phi \tag{3}$$

where Φ is the electric potential.

In terms of the three-dimensional metric h_{ij} the field equations following from the action (1) are:

$$\begin{aligned} {}^3R_{ij} = 2D_iuD_ju + 2D_i\varphi D_j\varphi - 2e^{-2u-2\alpha\varphi}D_i\Phi D_j\Phi \\ D_iD^iu = e^{-2u-2\alpha\varphi}D_i\Phi D^i\Phi \\ D_iD^i\varphi = \alpha e^{-2u-2\alpha\varphi}D_i\Phi D^i\Phi \\ D_i\left(e^{-2u-2\alpha\varphi}D^i\Phi\right) = 0\,. \end{aligned} \tag{4}$$

Here D_i is Levi-Civita connection and ${}^3R_{ij}$ is the Ricci tensor with respect to the three-metric h_{ij}.

Let us introduce the following symmetric matrix

$$\overset{\alpha}{S} = \begin{pmatrix} e^{(1+\alpha)u+(\alpha-1)\varphi} - (1+\alpha^2)\Phi^2 e^{(\alpha-1)u-(1+\alpha)\varphi} & -\sqrt{1+\alpha^2}\Phi e^{(\alpha-1)u-(1+\alpha)\varphi} \\ -\sqrt{1+\alpha^2}\Phi e^{(\alpha-1)u-(1+\alpha)\varphi} & -e^{(\alpha-1)u-(1+\alpha)\varphi} \end{pmatrix}. \tag{5}$$

In the terms of the matrix $\overset{\alpha}{S}$ the equations (4) can be written in the following compact form:

$$\begin{aligned} {}^3R_{ij} = -\frac{1}{1+\alpha^2}Sp\left(\partial_i\overset{\alpha}{S}\partial_j\overset{\alpha}{S}^{-1}\right) \\ D^i\left(\overset{\alpha}{S}^{-1}D_i\overset{\alpha}{S}\right) = 0\,. \end{aligned} \tag{6}$$

The above equations can be derived from the following action

$$\tilde{\mathcal{A}} = \int \sqrt{h}\left({}^3R - \frac{h^{ij}}{1+\alpha^2}Sp\left(D_i\overset{\alpha}{S}D_j\overset{\alpha}{S}^{-1}\right)\right)d^3x\,. \tag{7}$$

The action (7) is invariant under the group $GL(2,R)$ for fixed projection metric h_{ij}. The group $GL(2,R)$ acts explicitly as follows

$$\overset{\alpha}{S} \to G\overset{\alpha}{S}G^T$$

where $G \in GL(2,R)$.

The $GL(2,R)$ symmetry can be employed for generating new exact EMD solutions from known ones. In particular, it will be useful to employ the $GL(2,R)$ symmetry to generate exact solutions with nontrivial electric field from any given

solution of the pure vacuum Einstein equations or Einstein-dilaton equations (for details see [3]). The action of the symmetry group does not, in general, preserve the asymptotic flatness. The subgroup preserving the asymptotic flatness is $SO(1,1)$ (see [4]).

Another method which allows us to generate large classes of exact EMD solutions from solutions of pure vacuum Einstein equations will be discussed below.

We assume now that the matrix $\overset{\alpha}{S}$ depends on the space coordinates only through one potential χ satisfying the equation

$$D_i D^i \chi = 0\,. \tag{8}$$

Then requiring also that

$$-\frac{1}{2\,(1+\alpha^2)} Sp\left(\frac{d\overset{\alpha}{S}}{d\chi}\frac{d\overset{\alpha}{S}{}^{-1}}{d\chi}\right) = 1 \tag{9}$$

the equations (6) are reduced to the following

$$\begin{aligned} {}^3R_{ij} &= 2D_i\chi D_j\chi \\ D_i D^i \chi &= 0 \\ \frac{d}{d\chi}\left(\overset{\alpha}{S}{}^{-1}\frac{d}{d\chi}\overset{\alpha}{S}\right) &= 0\,. \end{aligned} \tag{10}$$

The important observation is that the first two equations of (10) are actually the static vacuum Einstein equations. The third equation is separated from the first two and can be formally integrated. Its general asymptotically flat solution is

$$\overset{\alpha}{S} = \sigma_3\, e^{Q\,\chi} \tag{11}$$

where $Q = \begin{pmatrix} a & b \\ -b & c \end{pmatrix}$ and σ_3 is the third Pauli matrix. In dependence of the determinant of the matrix $\overset{\alpha}{S}$ we obtain three classes of solutions.

The first class solutions is obtained when $\det Q < 1+\alpha^2$:

$$e^{2u} = (1-\Gamma^2)^{\frac{2}{1+\alpha^2}} \frac{e^{2\chi\cos(\omega-\omega_\alpha)}}{\left(1-\Gamma^2 e^{2\chi\sqrt{1+\alpha^2}\cos(\omega)}\right)^{\frac{2}{1+\alpha^2}}}\,,$$

$$e^{2\varphi} = (1-\Gamma^2)^{\frac{2\alpha}{1+\alpha^2}} \frac{e^{2\chi\sin(\omega-\omega_\alpha)}}{\left(1-\Gamma^2 e^{2\chi\sqrt{1+\alpha^2}\cos(\omega)}\right)^{\frac{2\alpha}{1+\alpha^2}}}\,,$$

$$\Phi = \frac{\Gamma}{\sqrt{1+\alpha^2}} \frac{1 - e^{2\chi\sqrt{1+\alpha^2}\cos(\omega)}}{1 - \Gamma^2 e^{2\chi\sqrt{1+\alpha^2}\cos(\omega)}} ,$$

$$\omega_\alpha = \arcsin\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right) .$$

The second class solutions is obtained for $\det Q = 1 + \alpha^2$:

$$e^{2u} = \frac{e^{\frac{2\alpha}{\sqrt{1+\alpha^2}}\chi}}{(1 - b\chi)^{\frac{2}{1+\alpha^2}}} ,$$

$$e^{2\varphi} = \frac{e^{-\frac{2}{\sqrt{1+\alpha^2}}\chi}}{(1 - b\chi)^{\frac{2\alpha}{1+\alpha^2}}} ,$$

$$\Phi = -\frac{1}{\sqrt{1+\alpha^2}} \frac{b\chi}{1 - b\chi} .$$

The third class solutions is obtained when $\det Q > 1 + \alpha^2$:

$$e^{2u} = e^{\frac{2\alpha\chi}{\sqrt{1+\alpha^2}}\cosh(\psi)} \frac{\cos^{\frac{2}{1+\alpha^2}}(\vartheta)}{\cos^{\frac{2}{1+\alpha^2}}(\chi\sqrt{1+\alpha^2}\sinh(\psi) + \vartheta)}$$

$$e^{2\varphi} = e^{-\frac{2\chi}{\sqrt{1+\alpha^2}}\cosh(\psi)} \frac{\cos^{\frac{2\alpha}{1+\alpha^2}}(\vartheta)}{\cos^{\frac{2\alpha}{1+\alpha^2}}(\chi\sqrt{1+\alpha^2}\sinh(\psi) + \vartheta)}$$

$$\Phi = -\frac{1}{\sqrt{1+\alpha^2}} \frac{\sin(\chi\sqrt{1+\alpha^2}\sinh(\psi))}{\cos(\chi\sqrt{1+\alpha^2}\sinh(\psi) + \vartheta)} .$$

Here the free parameters $\Gamma^2 < 1$, ω, Ψ and ϑ are functions of the original parameters a, b and c.

The obtained results can be summarized in the following

Theorem 1 *Let g^E is a static, asymptotically flat solution to the Einstein equations and $e^{2\chi} = -g^E(\xi, \xi)$. Then the metric*

$$g = e^{2\chi - 2u} g^E - \left(e^{-2u} - e^{2u - 4\chi}\right) \xi \otimes \xi$$

together with the two form $F = e^{-2u}\xi \wedge d\Phi$ and scalar field φ is a static and asymptotically flat solution to the EMD-gravity equations when $\mathbf{u}$, φ and Φ are given by one the classes (1), (2) and (3).

For spherically symmetric space-times most of the obtained solutions describe globally naked strong curvature singularities [2]. Only within the first class solutions, for the particular case $\omega = \omega_\alpha$, we obtain the charged dilaton black hole solution.

It is also interesting to consider static, axi-symmetric space-times. In this case the metric can be written in the form

$$ds^2 = -e^{2u}dt^2 + e^{2h-2u}(d\rho^2 + dz^2) + e^{-2u}\rho^2 d\phi^2 \,. \tag{12}$$

For static, axi-symmetric space-times the EMD equations take the form

$$\begin{gathered}
\partial_\rho \left(\rho \overset{\alpha}{S}{}^{-1} \partial_\rho \overset{\alpha}{S} \right) + \partial_z \left(\rho \overset{\alpha}{S}{}^{-1} \partial_z \overset{\alpha}{S} \right) = 0 \\
\frac{1}{\rho}\partial_\rho h = \frac{1}{2(1+\alpha^2)} \left(Sp \left(\partial_z \overset{\alpha}{S}{}^{-1} \partial_z \overset{\alpha}{S} \right) - Sp \left(\partial_\rho \overset{\alpha}{S}{}^{-1} \partial_\rho \overset{\alpha}{S} \right) \right) \\
\frac{1}{\rho}\partial_z h = -\frac{1}{1+\alpha^2} Sp \left(\partial_\rho \overset{\alpha}{S} \partial_z \overset{\alpha}{S}{}^{-1} \right) \,.
\end{gathered} \tag{13}$$

The first equation of the above system is well-known. This equation can be solved by use of the inverse scattering problem method. This method allows us to construct n-soliton solutions. The inverse scattering problem method, however, requires tedious calculations. It seems that the method described above is more powerful than the inverse scattering problem method and it gives larger classes of solutions in more direct and simpler manner. Assuming again that the matrix $\overset{\alpha}{S}$ depends on the space-coordinates only through the one harmonic potential χ we obtain

$$\overset{\alpha}{S} = \sigma_3 \, e^{Q\,\chi}$$

where

$$\partial_z^2 \chi + \frac{1}{\rho}\partial_\rho \chi + \partial_\rho^2 \chi = 0 \,.$$

Note that here the elements of the matrix Q are not constrained by (9) and can be arbitrary.

We would also like to discuss briefly once more method for constructing exact EMD solutions. One may wonder whether exact EMD solutions can be constructing from solutions of Einstein-Maxwell equations. We shall demonstrate that this is possible [3], [4].

Let us go back to the static, axisymmetric EMD equations (13) and to introduce the new potentials $U = u + \alpha\varphi$, $\Psi = \alpha u - \varphi$, $\Lambda = \sqrt{1+\alpha^2}\Phi$ and the new metric function $H = (1+\alpha^2)h$. The static, axisymmetric EMD equations can be rewritten in the form

$$
\begin{aligned}
\partial_\rho^2\Psi + \frac{1}{\rho}\partial_\rho\Psi + \partial_z^2\Psi = 0 \\
\partial_\rho^2 U + \frac{1}{\rho}\partial_\rho U + \partial_z^2 U = e^{-2U}\left((\partial_\rho\Lambda)^2 + (\partial_z\Lambda)^2\right) \\
\partial_\rho\left(\rho e^{-2U}\partial_\rho\Lambda\right) + \partial_z\left(\rho e^{-2U}\partial_z\Lambda\right) = 0 \\
\frac{1}{\rho}\partial_\rho H = (\partial_\rho U)^2 - (\partial_z U)^2 + (\partial_\rho\Psi)^2 - (\partial_z\Psi)^2 - e^{-2U}\left((\partial_\rho\Lambda)^2 - (\partial_z\Lambda)^2\right) \\
\frac{1}{\rho}\partial_z H = 2\partial_\rho U\partial_z U + 2\partial_\rho\Psi\partial_z\Psi - 2e^{-2U}\partial_\rho\Lambda\partial_z\Lambda\,.
\end{aligned}
\tag{14}
$$

It is not difficult to recognize that the above system is just the static, axisymmetric Einstein-Maxwell equations along together with minimally coupled scalar field Ψ. Therefore we obtain

Theorem 2 *Let U, Ψ, H and Λ form a solution of the static, axisymmetric Einstein-Maxwell equations with minimally coupled scalar field. Then $u = (1+\alpha^2)^{-1}(U + \alpha\Psi)$, $\varphi = (1+\alpha^2)^{-1}(\alpha U - \Psi)$, $h = (1+\alpha^2)^{-1}H$ and $\Phi = (1+\alpha^2)^{-1/2}\Lambda$ form a solution of static, axisymmetric EMD equations.*

The line of the present investigation has been continued for the EMD cosmological space-times in [3].

Another approach to finding exact EMD solutions can be found in [5], [6], [7].

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